

A period differential equation for a family of $K3$ surfaces and the Hilbert modular orbifold for the field $\mathbb{Q}(\sqrt{5})$

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Abstract

In this article we study the period map for a family of $K3$ surfaces which is given by the anticanonical divisor of a toric variety. We determine the period differential equation and its monodromy group. Moreover we show the exact relation between our period differential equation and the uniformizing differential equation of the Hilbert modular orbifold for the field $\mathbb{Q}(\sqrt{5})$.

Introduction

The elliptic modular function $\lambda(\tau)$ is obtained as the inverse of the Schwarz map for the Gauss hypergeometric differential equation

$$\lambda(1-\lambda)\frac{d^2\eta}{d\lambda^2} + (1-2\lambda)\frac{d\eta}{d\lambda} - \frac{1}{4}\eta = 0.$$

This is a period differential equation for the family of elliptic curves $\mathcal{F} = \{S(\lambda)\}$, with

$$S(\lambda) : y^2 = x(x-1)(x-\lambda).$$

A $K3$ surface is characterized as a deformation of a non singular quartic surface in $\mathbb{P}^3(\mathbb{C})$. It suggests that a $K3$ surface is a 2-dimensional analogy of an elliptic curve (namely a non singular cubic curve in $\mathbb{P}^2(\mathbb{C})$). There are several studies on $K3$ modular functions, for example [MSY], [Sh1], [Sh2] and [Sh3]. Period differential equations play important roles in these studies. On the other hand, a $K3$ surface S is characterized by the condition $K_S = 0$ and simply connectedness. It means that S is a 2-dimensional Calabi-Yau manifold. There are many studies of Calabi-Yau manifolds in the field of mathematical physics. From this aspects, the inverse of the Schwarz map is called the "mirror map".

We use the notion of the 3-dimensional reflexive polytope (it is due to Batyrev [Ba]) with at most terminal singularities. Such a polytope P is defined by the intersection of several half spaces

$$a_j x + b_j y + c_j z \leq 1, (a_j, b_j, c_j) \in \mathbb{Z}^3, j = 1, \dots, s$$

in \mathbb{R}^3 with the conditions

- (i) the origin is the unique inner lattice point,
- (ii) only the vertices are the lattice points on the boundary.

For every such a reflexive polytope we can find a corresponding family of $K3$ surfaces. Period integrals for such families of $K3$ surfaces are studied by K. Koike [Koi], J. Stienstra [St], N. Narumiya and H. Shiga [NS].

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All 3-dimensional 5-verticed reflexive polytopes with at most terminal singularity are listed up (see [KS] or [O]). Recently T. Ishige [I2] has made a detailed research on one of these polytopes. He discovered, by a computer approximation, the projective monodromy group of his period differential equation is isomorphic to the extended Hilbert modular group for the field $\mathbb{Q}(\sqrt{2})$.

Inspired by Ishige's discovery we have studied these families of $K3$ surfaces derived from other polytopes. In this article we focus on the polytope

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}, \quad (0.1)$$

where the column vectors indicate the vertices. We study the other polytopes in another article [N].

In this article we construct an exact period map Φ for the family, saying \mathcal{F} , of our $K3$ surfaces, that is a multivalued analytic map from a Zariski open domain Λ in $\mathbb{P}^2(\mathbb{C})$ to a 2-dimensional domain of type IV . We give a differential equation for Φ and determine its projective monodromy group. Moreover we show the exact relation between our period differential equation and the Hilbert modular orbifold for the real quadratic field $\mathbb{Q}(\sqrt{5})$.

In Section 1 we show the explicit defining equations for the family, saying \mathcal{F} , of $K3$ surfaces derived from the polytope P in (0.1). In Section 2 we show that the Picard number of a generic member of \mathcal{F} is equal to 18 (Theorem 2.2) and determine the structure of the Néron-Severi lattice and the transcendental lattice (Theorem 2.3).

According to Theorem 2.2 there is a period differential equation with 4-dimensional solution space. We determine it in Section 3 (Theorem 3.2). In Section 4, applying the Torelli theorem and lattice theory, we prove that the projective monodromy group of the period map is equal to the orthogonal group for the transcendental lattice (Theorem 4.2). This is essentially a complete proof of the above mentioned Ishige's discovery.

Theorem 4.2 implies that the projective monodromy group of the period differential equation for \mathcal{F} is related to the Hilbert modular group of the field $\mathbb{Q}(\sqrt{5})$. On the other hand the uniformizing differential equation of the Hilbert modular orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(\mathcal{O}), \tau \rangle$ is studied by T. Sato [Sa]. In Section 5 we show an explicit correspondence of our period differential equation and the uniformizing differential equation of the Hilbert modular orbifold.

1 Family of $K3$ surfaces and elliptic fibration

Let us start from the polytope P in (0.1). We obtain a family of algebraic surfaces by the following canonical procedure:

- (i) Make a toric 3-fold X from the reflexive polytope P . This is a rational variety.
- (ii) Take a divisor D on X that is linearly equivalent to $-K_X$.
- (iii) Generically D is represented by a $K3$ surface.

In this case, D is given by

$$a_1 + a_2 t_1 + a_3 t_2 + a_4 t_3 + a_5 \frac{1}{t_3} + a_6 \frac{1}{t_1 t_2 t_3^2} = 0, \quad (1.1)$$

with complex parameters a_1, \dots, a_6 . Namely, P plays the same role as the Newton polygon. Setting

$$\begin{cases} x = a_2 t_1 / a_1, \\ y = a_3 t_2 / a_1, \\ z = a_4 t_3 / a_1, \\ \lambda = a_4 a_5 / a_1^2, \\ \mu = a_2 a_3 a_4^2 a_6 / a_1^5, \end{cases} \quad (1.2)$$

from (1.1) we obtain a family of $K3$ surfaces $\mathcal{F} = \{S(\lambda, \mu)\}$ with two parameters λ, μ with

$$S(\lambda, \mu) : xyz^2(x + y + z + 1) + \lambda xyz + \mu = 0. \quad (1.3)$$

We can find an elliptic fibration for every surface of our family \mathcal{F} . Namely it can be described in the form

$$S : y^2 = 4x^3 - g_2(z)x_3 - g_3(z),$$

where g_2 (g_3 , resp.) is a polynomial of z with $5 \leq \deg(g_2) \leq 8$ ($7 \leq \deg(g_3) \leq 12$, resp.). This is an analogy of the Weierstrass normal form of the elliptic curve. In this paper we call it the Kodaira normal form.

Proposition 1.1. $S(\lambda, \mu) \in \mathcal{F}$ is described in the Kodaira normal form

$$y_1^2 = 4x_1^3 - g_2(z)x_1 - g_3(z), z \neq \infty, \quad (1.4)$$

with

$$\begin{cases} g_2(z) = \frac{1}{216}(18\lambda^4 + 432\lambda\mu z + 72\lambda^3 z(1+z) + 108\lambda^2 z^2(1+z)^2 \\ \quad + 72\lambda z^3(1+z)^3 + 18z^2(1+z)(24\mu + z^2(1+z)^3), \\ g_3(z) = \frac{-1}{216}(\lambda^6 + 36\lambda^3\mu z + 6\lambda^5 z(1+z) + 108\lambda^2\mu z^2(1+z) + 15\lambda^4 z^2(1+z)^2 \\ \quad + 108\lambda\mu z^3(1+z)^2 + 20\lambda^3 z^3(1+z)^3 + 15\lambda^2 z^4(1+z)^4 + 6\lambda z^5(1+z)^5 \\ \quad + z^2(216\mu^2 + 36\mu z^2(1+z)^3 + z^4(1+z)^6)), \end{cases}$$

and

$$y_2^2 = 4x_2^3 - h_2(z_1)x_2 - h_3(z_1), z_1 \neq \infty, \quad (1.5)$$

with

$$\begin{cases} h_2(z_1) = 2\mu z_1^5(1 + z_1 + \lambda z_1^2) + \frac{1}{12}(1 + z_1 + \lambda z_1^2)^4, \\ h_3(z_1) = -(\frac{1}{6}\mu z_1^5(1 + z_1 + \lambda z_1^2)^3 + \frac{1}{216}(1 + z_1 + \lambda z_1^2)^6 + \mu^2 z_1^{10}), \end{cases}$$

where $z_1 = 1/z$. We have the discriminant of the right hand side for x_1 (x_2 , resp.):

$$\begin{cases} D_0 = 64\mu^3 z^3(\lambda^3 + 3\lambda^2 z + 27\mu z + 3\lambda z^2 + 3\lambda^2 z^2 + z^3 + 6\lambda z^3 + 3z^4 + 3\lambda z^4 + 3z^5 + z^6), \\ D_\infty = 64\mu^3 z_1^{15}(1 + 3z_1 + 3z_1^2 + 3\lambda z_1^2 + z_1^3 + 6\lambda z_1^3 + 3\lambda z_1^4 + 3\lambda^2 z_1^4 + 3\lambda^2 z_1^5 + 27\mu z_1^5 + \lambda^3 z_1^6), \end{cases} \quad (1.6)$$

respectively.

Proof. By the birational transformation

$$x_0 = \frac{-\mu}{x}, \quad y_1 = \frac{\mu}{x^2 z}(2xyz^2 + (\lambda xz + xz^2 + x^2 z^2 + xz^3))$$

the equation (1.3) takes the form

$$y_1^2 = 4x_0^3 + (\lambda^2 + 2\lambda z + z^2 + 2\lambda z^2 + 2z^3 + z^4)x_0^2 + (-2\lambda\mu z - 2\mu z^2 - 2\mu z^3)x_0 + \mu^2 z^2. \quad (1.7)$$

Moreover, by the transformation

$$x_1 = x_0 + \frac{1}{12}(\lambda^2 + 2\lambda\mu + z^2 + 2\lambda z^2 + 2z^3 + z^4),$$

we obtain (1.4).

Set $z_1 = 1/z$. By the transformation

$$x_2 = x_0 z_1^4 + \frac{1}{12}(1 + z_1 + \lambda z_1^2)^2, \quad y_2 = z_1^6 y_1,$$

we get (1.5). □

Set

$$\Lambda = \{(\lambda, \mu) \in \mathbb{C}^2 \mid \lambda\mu(\lambda^2(4\lambda - 1)^3 - 2(2 + 25\lambda(20\lambda - 1))\mu - 3125\mu^2) \neq 0\}.$$

We have a parametrization

$$\lambda(a) = \frac{(a-1)(a+1)}{5}, \quad \mu(a) = \frac{(2a-3)^3(a+1)^2}{3125}$$

of the locus $\lambda^2(4\lambda - 1)^3 - 2(2 + 25\lambda(20\lambda - 1))\mu - 3125\mu^2 = 0$. So it is a rational curve.

Remark 1.1. *In Section 3 we shall obtain Λ as the complement of the singular locus of the period differential equation for $S(\lambda, \mu)$ in the (λ, μ) -space.*

Proposition 1.2. *Suppose $(\lambda, \mu) \in \Lambda$. The elliptic surface given by (1.4) and (1.5) has singular fibres of type I_3 over $z = 0$, of type I_{15} over $z = \infty$ and other six fibres of type I_1 .*

Proof. By observing the orders of g_2, g_3, D_0 (h_2, h_3, D_∞ , resp.) we know the type of every singular fibre (see [Sh3]). \square

Remark 1.2. *Let χ denote the Euler characteristic. According to [Kod] (see also [Sh3]), an elliptic fibred algebraic surface S over $\mathbb{P}^1(\mathbb{C})$ is a K3 surface if and only if $\chi(S) = 24$ provided S is given in the Kodaira normal form. Using this criterion we can check directly from Proposition 1.2 that $S(\lambda, \mu)$ is a K3 surface for $(\lambda, \mu) \in \Lambda$.*

Proposition 1.3. *The elliptic surface given by (1.4) and (1.5) has the following holomorphic section:*

$$P : z \mapsto (x_1, y_1, z) = \left(\frac{1}{12}(\lambda^2 + 2\lambda z + z^2 + 2\lambda z^2 + 2z^3 + z^4), \mu z, z \right).$$

Proof. By putting $x_0 = 0$ in (1.7), it is obvious. \square

For $(\lambda, \mu) \in \Lambda$, let O be the zero of the Mordell-Weil group of $S(\lambda, \mu)$ over $\mathbb{C}(z)$. O is given by the set of the points at infinity on every fibre. Let Q be the inverse element of P in the Mordell-Weil group. Let $I_3 = a_0 + a_1 + a'_1$ be the irreducible decomposition of the fibre at $z = 0$. We may suppose $O \cap a_0 \neq \phi, P \cap a_1 \neq \phi, Q \cap a'_1 \neq \phi$. By the same way, let $I_{15} = b_0 + b_1 + \cdots + b_7 + b'_1 + \cdots + b'_7$ be the irreducible decomposition of the fibre at $z = \infty$ given by the Figure 2. We may suppose $O \cap b_0 \neq \phi, P \cap b_5 \neq \phi, Q \cap b'_5 \neq \phi$.

Figure of I3

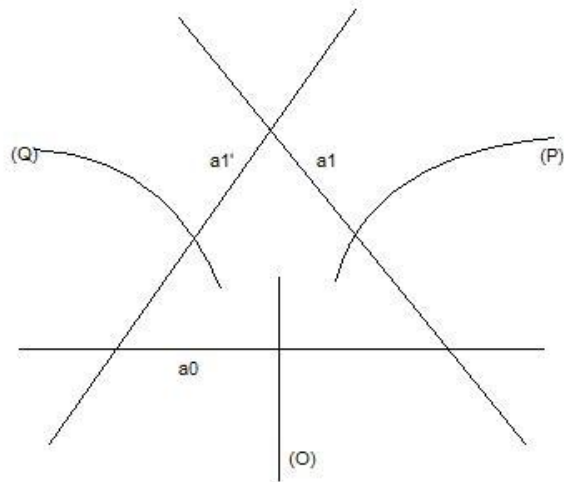


Figure 1.

Figure of I15

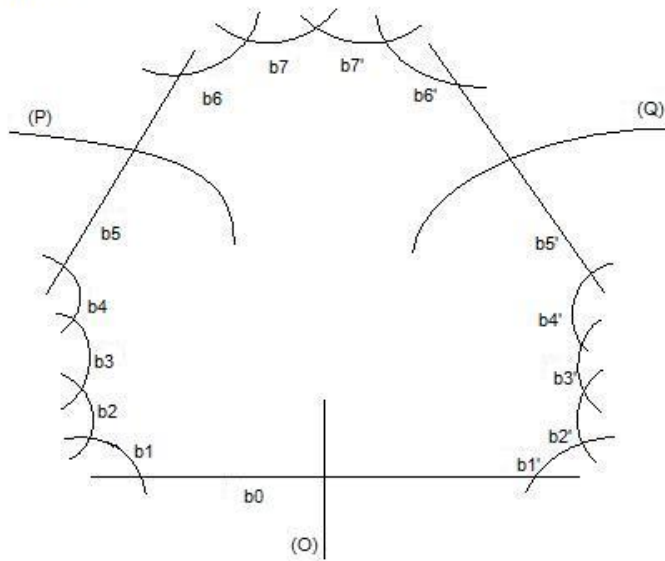


Figure 2.

For a general $K3$ surface S , $H_2(S, \mathbb{Z})$ is a free \mathbb{Z} -module of rank 22. The intersection form of $H_2(S, \mathbb{Z})$ is given by

$$E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U,$$

where

$$E_8(-1) = \begin{pmatrix} -2 & 1 & & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & 1 & -2 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & 1 & \\ & & & & & 1 & -2 & 0 & \\ & & & & & & 1 & 0 & -2 & 1 \\ & & & & & & & & 1 & -2 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\text{NS}(S)$ denote the sublattice in $H_2(S, \mathbb{Z})$ generated by the divisors on S .

We set a sublattice $L'(\lambda, \mu) \subset H_2(S(\lambda, \mu), \mathbb{Z})$ for $(\lambda, \mu) \in \Lambda$ by

$$L'(\lambda, \mu) = \langle b_1, b_2, b_3, b_4, b_5, P, b_6, b_7, b'_1, b'_2, b'_3, b'_4, b'_5, Q, b'_6, b'_7, O, F \rangle_{\mathbb{Z}}. \quad (1.8)$$

It is contained in $\text{NS}(S(\lambda, \mu))$ and of rank 18. So we have:

Proposition 1.4.

$$\text{rank } \text{NS}(S(\lambda, \mu)) \geq 18.$$

We have:

Proposition 1.5. $L'(\lambda, \mu)$ is a primitive sublattice of $H_2(S(\lambda, \mu), \mathbb{Z})$.

Proof. By observing the intersection matrix of the lattice $L'(\lambda, \mu)$, we obtain $\det(L'(\lambda, \mu)) = -5$. It does not contain any square factor. So $L'(\lambda, \mu)$ is primitive. \square

Definition 1.1. Let $(S_1, \pi_1, \mathbb{P}^1(\mathbb{C})), (S_2, \pi_2, \mathbb{P}^1(\mathbb{C}))$ be elliptic surfaces. If there exist a biholomorphic mapping $f : S_1 \rightarrow S_2$ and $\varphi \in \text{Aut}(\mathbb{P}^1(\mathbb{C}))$ such that $\varphi \circ \pi_1 = \pi_2 \circ f$, we say $(S_1, \pi_1, \mathbb{P}^1(\mathbb{C}))$ and $(S_2, \pi_2, \mathbb{P}^1(\mathbb{C}))$ are isomorphic as elliptic surfaces.

For an elliptic surface given by the Kodaira normal form $y^2 = 4x^3 - g_2(z)x - g_3(z)$, we can define the j -invariant :

$$j(z) = \frac{g_2^3(z)}{g_2^3(z) - 27g_3^2(z)} \in \mathbb{C}(z). \quad (1.9)$$

Proposition 1.6. ([Kod]) Let $(S_1, \pi_1, \mathbb{P}^1(\mathbb{C})), (S_2, \pi_2, \mathbb{P}^1(\mathbb{C}))$ be elliptic surfaces, and let $j_1(z), j_2(z)$ be the j -invariants, respectively. If $(S_1, \pi_1, \mathbb{P}^1(\mathbb{C})), (S_2, \pi_2, \mathbb{P}^1(\mathbb{C}))$ are isomorphic, it is necessary that there exists $\varphi \in \text{Aut}(\mathbb{P}^1(\mathbb{C}))$ such that $\pi_1^{-1}(p)$ and $\pi_2^{-1}(\varphi(p))$ are the fibres of the same type for any $p \in \mathbb{P}^1(\mathbb{C})$ and $j_2 \circ \varphi = j_1$.

For $(\lambda, \mu) \in \Lambda$, let

$$\pi : S(\lambda, \mu) \rightarrow \mathbb{P}^1(\mathbb{C}) = (z\text{-sphere})$$

be the canonical elliptic fibration given by the Kodaira normal form (1.4), (1.5). $(S(\lambda, \mu), \pi, \mathbb{P}^1)$ is an elliptic surface.

Lemma 1.1. Suppose $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Lambda$. If $(S(\lambda_1, \mu_1), \pi_1, \mathbb{P}^1(\mathbb{C}))$ is isomorphic to $(S(\lambda_2, \mu_2), \pi_2, \mathbb{P}^1(\mathbb{C}))$ as elliptic surfaces, then it holds $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.

Proof. Let $f : S_1 \rightarrow S_2$ be the biholomorphic isomorphism which gives the equivalence of elliptic surfaces. According to Proposition 1.6, there exists $\varphi \in \text{Aut}(\mathbb{P}^1(\mathbb{C}))$ which satisfies $\varphi \circ \pi_1 = \pi_2 \circ f$. By Proposition 1.2, $\pi_j^{-1}(0) = I_3$ and $\pi_j^{-1}(\infty) = I_{15}$ ($j = 1, 2$). So φ has the form $\varphi : z \mapsto az$ with some $a \in \mathbb{C} - 0$. Let $D_0(z; \lambda_j, \mu_j)$ ($j = 1, 2$) be the discriminant. From (1.6), we have

$$\frac{D_0(z; \lambda_j, \mu_j)}{64\mu_j^3 z^3} = \lambda_j^3 + 3\lambda_j^2 z + 27\mu_j z + 3\lambda_j z^2 + 3\lambda_j^2 z^2 + z^3 + 6\lambda_j z^3 + 3z^4 + 3\lambda_j z^4 + 3z^5 + z^6,$$

$j = 1, 2$. The six roots of $D_0(z; \lambda_1, \mu_1)/64\mu_1^3 z^3$ ($D_0(z; \lambda_2, \mu_2)/64\mu_2^3 z^3$, resp.) give the six images of singular fibres of type I_1 of $S(\lambda_1, \mu_1)$ ($S(\lambda_2, \mu_2)$, resp.). The roots of $D_0(z; \lambda_1, \mu_1)/64\mu_1^3 z^3$ are sent by φ to those of $D_0(z; \lambda_2, \mu_2)/64\mu_2^3 z^3$. Observing the coefficients of $D_0(z; \lambda_1, \mu_1)$ and $D_0(z; \lambda_2, \mu_2)$, we obtain that $a = 1$ and $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$. \square

2 Period map, the Picard number and the Néron-Severi lattice for \mathcal{F}

In this section, we define the period map for the family \mathcal{F} and determine the Picard number and the lattice structure of a generic member of \mathcal{F} .

2.1 S-marked K3 surfaces

Definition 2.1. Let $S_0 = S(\lambda_0, \mu_0)$ be a reference surface for a fixed point $(\lambda_0, \mu_0) \in \Lambda$. For a K3 surface $S(\lambda, \mu)$ ($(\lambda, \mu) \in \Lambda$), set

$$\begin{aligned} L'(\lambda, \mu) &= \langle b_1, b_2, b_3, b_4, b_5, P, b_6, b_7, b'_1, b'_2, b'_3, b'_4, b'_5, Q, b'_6, b'_7, O, F \rangle_{\mathbb{Z}} \\ &= \langle \gamma_5, \dots, \gamma_{22} \rangle_{\mathbb{Z}} \subset H_2(S(\lambda, \mu), \mathbb{Z}), \end{aligned}$$

given by (1.8). Set $L = H_2(S_0, \mathbb{Z})$ and $L' = L'(\lambda_0, \mu_0)$. We define a S -marking ψ to be an isomorphism $\psi : H_2(S(\lambda, \mu), \mathbb{Z}) \rightarrow L$ with the property that $\psi^{-1}(\gamma_j) = \gamma_j$ for $5 \leq j \leq 22$. We call the pair $(S(\lambda, \mu), \psi)$ an S -marked K3 surface.

By definition, ψ has the following property:

$$\begin{aligned} \psi^{-1}(F) &= F, \psi^{-1}(O) = O, \psi^{-1}(P) = P, \psi^{-1}(Q) = Q, \\ \psi^{-1}(b_j) &= b_j, \psi^{-1}(b'_j) = b'_j \end{aligned}$$

Definition 2.2. Two S -marked K3 surfaces (S, ψ) and (S', ψ') are said to be isomorphic if we have a biholomorphic map $f : S \rightarrow S'$ with

$$\psi' \circ f_* \circ \psi^{-1} = \text{id}_L.$$

Two S -marked K3 surfaces (S, ψ) and (S', ψ') are said to be equivalent if we have a biholomorphic map $f : S \rightarrow S'$ with

$$\psi' \circ f_* \circ \psi^{-1}|_{L'} = \text{id}_{L'}.$$

By Proposition 1.5, the basis $\{\gamma_5, \dots, \gamma_{22}\}$ of L' can be extended to a basis

$$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \dots, \gamma_{22}\} \quad (2.1)$$

of L . Let $\{\gamma_1^*, \dots, \gamma_{22}^*\}$ be a dual basis of $\{\gamma_1, \dots, \gamma_{22}\}$ with respect to the intersection form, namely $(\gamma_j \cdot \gamma_k^*) = \delta_{j,k}$. Set

$$L'' = \langle \gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^* \rangle_{\mathbb{Z}} \subset L. \quad (2.2)$$

We have $L'' = L'^{\perp}$.

2.2 Period map

First we state the definition of the period map for general $K3$ surfaces.

For a $K3$ surface S , there exists unique holomorphic 2-form ω up to a constant factor. Let $\{\gamma_1, \dots, \gamma_{22}\}$ be a basis of $H_2(S, \mathbb{Z})$.

$$\eta' = \left(\int_{\gamma_1} \omega : \dots : \int_{\gamma_{22}} \omega \right) \in \mathbb{P}^{21}(\mathbb{C})$$

is said to be a period of S . The Néron-Severi lattice $\text{NS}(S)$ is defined as the sublattice of $H_2(S, \mathbb{Z})$ generated by the divisors on S . $\text{Tr}(S) = \text{NS}(S)^\perp$ is said to be the transcendental lattice of S . Let $\{\gamma_1, \dots, \gamma_r\}$ be a basis of $\text{Tr}(S)$. Note

$$\int_{\gamma} \omega = 0, \quad (\forall \gamma \in \text{NS}(S)). \quad (2.3)$$

So the period η' reduces to

$$\eta = \left(\int_{\gamma_1} \omega : \dots : \int_{\gamma_r} \omega \right) \in \mathbb{P}^{r-1}(\mathbb{C}).$$

We note that $\text{NS}(S)$ is a lattice of signature $(1, \cdot)$ and $\text{Tr}(S)$ is a lattice of the signature $(2, \cdot)$.

We define the period map for our case in the following way.

Definition 2.3. Let S_0 be the above reference surface. Take a small neighborhood δ of (λ_0, μ_0) in Λ so that we have a local topological trivialization

$$\tau : \{S(\lambda, \mu) : (\lambda, \mu) \in \delta\} \rightarrow S_0 \times \delta.$$

Take an S -marking ψ_0 of S_0 , and define the S -markings of $S(\lambda, \mu)$ by $\psi = \psi_0 \circ \tau_*$ for $(\lambda, \mu) \in \delta$. We define the local period map $\Phi : \delta \rightarrow \mathbb{P}^3(\mathbb{C})$ by

$$\Phi((\lambda, \mu)) = \left(\int_{\psi^{-1}(\gamma_1)} \omega : \dots : \int_{\psi^{-1}(\gamma_4)} \omega \right), \quad (2.4)$$

where $\gamma_1, \dots, \gamma_4 \in L$ are given by (2.1). We define the multivalued period map $\Lambda \rightarrow \mathbb{P}^3(\mathbb{C})$ by making the analytic continuation of Φ along any arc starting from (λ_0, μ_0) in Λ .

In general, we have the Riemann-Hodge relation for the period:

$$\begin{cases} \eta' M^t \eta' = 0, \\ \eta' M^t \bar{\eta}' > 0, \end{cases}$$

where M is the intersection matrix $(\gamma_j^* \cdot \gamma_k^*)_{1 \leq j, k \leq 22}$.

By considering the relation (2.3), in our situation, the Riemann-Hodge relation is reduced to

$$\begin{cases} \eta A^t \eta = 0, \\ \eta A^t \bar{\eta} > 0, \end{cases} \quad (2.5)$$

where

$$A = (\gamma_j^* \cdot \gamma_k^*)_{1 \leq j, k \leq 4}$$

and

$$\eta = \left(\int_{\psi^{-1}(\gamma_1)} \omega : \int_{\psi^{-1}(\gamma_2)} \omega : \int_{\psi^{-1}(\gamma_3)} \omega : \int_{\psi^{-1}(\gamma_4)} \omega \right).$$

Remark 2.1. We shall show (in Theorem 2.3) that the above matrix A is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Set

$$\mathcal{D} = \{\xi = (\xi_1 : \xi_2 : \xi_3 : \xi_4) \in \mathbb{P}^3(\mathbb{C}) \mid \xi A^t \xi = 0, \xi A^t \bar{\xi} > 0\}.$$

We have $\Phi(\Lambda) \subset \mathcal{D}$. Note that \mathcal{D} is composed of two connected components \mathcal{D}^+ and \mathcal{D}^- .

Definition 2.4. *The fundamental group $\pi_1(\Lambda, *)$ acts on the \mathbb{Z} -module $\langle \psi^{-1}(\gamma_1), \dots, \psi^{-1}(\gamma_4) \rangle_{\mathbb{Z}}$, so it induces the action on \mathcal{D} . This action induces a group of projective linear transformations which is a subgroup of $PGL(4, \mathbb{Z})$. We call it the projective monodromy group of the period map $\Phi : \Lambda \rightarrow \mathbb{P}^3(\mathbb{C})$.*

2.3 The Picard number

By the Riemann-Roch Theorem for surfaces and the Serre duality, we have the following lemma.

Lemma 2.1. *Let S be a K3 surface with elliptic fibration $\pi : S \rightarrow \mathbb{P}^1(\mathbb{C})$, and let F be a fixed general fibre. Then π is the unique elliptic fibration up to $\text{Aut}(\mathbb{P}^1(\mathbb{C}))$ which has F as a general fibre.*

We can formulate the injectivity of the Torelli theorem for S-marked K3 surfaces in its local version:

Theorem 2.1. *Let $\delta \subset \Lambda$ be a sufficiently small neighborhood of (λ_0, μ_0) , and $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \delta$. Suppose $\Phi(\lambda_1, \mu_1) = \Phi(\lambda_2, \mu_2)$, then there exists an isomorphism of S-marked K3 surfaces $(S(\lambda_1, \mu_1), \psi_1) \simeq (S(\lambda_2, \mu_2), \psi_2)$.*

We determine the Picard number of a generic member of $\mathcal{F} = \{S(\lambda, \mu)\}$.

Proposition 2.1. *Two S-marked K3 surfaces $(S(\lambda_1, \mu_1), \psi_1), (S(\lambda_2, \mu_2), \psi_2)$ are equivalent if and only if there exists an isomorphism of elliptic surfaces $(S(\lambda_1, \mu_1), \pi_1, \mathbb{P}^1(\mathbb{C})) \simeq (S(\lambda_2, \mu_2), \pi_2, \mathbb{P}^1(\mathbb{C}))$.*

Proof. The sufficiency is obvious. We prove the necessity. Set $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Lambda$. Suppose the equivalence of S-marked K3 surfaces

$$(S(\lambda_1, \mu_1), \psi_1) \simeq (S(\lambda_2, \mu_2), \psi_2).$$

Then there is a biholomorphic map $f : S(\lambda_1, \mu_1) \rightarrow S(\lambda_2, \mu_2)$ such that $\psi_2 \circ f_* \circ \psi_1^{-1}|_{L'} = id_{L'}$. Especially, for general fibres $F_1 \in \text{Div}(S_1)$ and $F_2 \in \text{Div}(S_2)$, we have $f_*(F_1) = F_2$.

So S_2 has two elliptic fibrations π_2 and $\pi_1 \circ f^{-1}$ which have a general fibre F_2 . According to Lemma 2.1, it holds

$$\pi_2 = \pi_1 \circ f^{-1}$$

up to $\text{Aut}(\mathbb{P}^1(\mathbb{C}))$. □

Corollary 2.1. *Let (λ_1, μ_1) and (λ_2, μ_2) be in Λ . Two S-marked K3 surfaces $(S(\lambda_1, \mu_1))$ and $(S(\lambda_2, \mu_2), \mu_2)$ are equivalent if and only if $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.*

Proof. From the proposition and Lemma 1.1, we have the required statement. □

Theorem 2.2. *For a generic point $(\lambda, \mu) \in \Lambda$, we have*

$$\text{rank NS}(S(\lambda, \mu)) = 18.$$

Proof. By Proposition 1.4 we already have $\text{rank NS}(S(\lambda, \mu)) \geq 18$. Let δ be a small neighborhood of (λ, μ) . Suppose we have $\text{rank NS}(S(\lambda', \mu')) > 18$ for all $(\lambda', \mu') \in \delta$. Then $\Phi(\delta)$ cannot contain any open set of \mathcal{D} . By Theorem 2.1 and Corollary 2.1, the period map is injective. This is a contradiction. □

Corollary 2.2. *The \mathbb{C} -vector space generated by the germs of holomorphic functions*

$$\int_{\psi^{-1}(\gamma_1)} \omega, \dots, \int_{\psi^{-1}(\gamma_4)} \omega$$

is 4-dimensional.

Proof. It is obvious because the rank of transcendental lattice $\text{Tr}(S(\lambda, \mu))$ is $22 - 18 = 4$. □

2.4 Lattice structure

Let (λ, μ) be a generic element of Λ . According to Proposition 1.5 and Theorem 2.2, we have a basis of $NS(S(\lambda, \mu))$:

$$\{b_1, b_2, b_3, b_4, b_5, P, b_6, b_7, b'_1, b'_2, b'_3, b'_4, b'_5, Q, b'_6, b'_7, F, O\}. \quad (2.6)$$

We obtain the corresponding intersection matrix:

[illegible]

Theorem 2.3. *For a generic point $(\lambda, \mu) \in \Lambda$, the intersection matrix of $\text{NS}(S(\lambda, \mu))$ is given by*

$$M_0 = \begin{pmatrix} E_8(-1) & & & \\ & E_8(-1) & & \\ & & 2 & 1 \\ & & 1 & -2 \end{pmatrix}, \quad (2.8)$$

and that of $\text{Tr}(S(\lambda, \mu))$ is given by

$$A = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 2 & 1 \\ & & 1 & -2 \end{pmatrix}. \quad (2.9)$$

Proof. Let M_1 be the matrix described in (2.7). Let U be the unimodular matrix

$$\begin{pmatrix} 1 & & & & & & & & & & & & & & & & -1 & 5 & -2 \\ & 1 & & & & & & & & & & & & & & & -2 & 10 & -4 \\ & & 1 & & & & & & & & & & & & & & -3 & 15 & -6 \\ & & & 1 & & & & & & & & & & & & & -4 & 20 & -8 \\ & & & & 1 & & & & & & & & & & & & -5 & 25 & -10 \\ & & & & & 1 & & & & & & & & & & & -2 & 13 & -6 \\ & & & & & & 1 & & & & & & & & & & -4 & 17 & -6 \\ & & & & & & & 1 & & & & & & & & & -3 & 9 & -2 \\ & & & & & & & & 1 & & & & & & & & -1 & 1 & 0 \\ & & & & & & & & & 1 & & & & & & & -2 & 2 & 0 \\ & & & & & & & & & & 1 & & & & & & -3 & 3 & 0 \\ & & & & & & & & & & & 1 & & & & & -4 & 4 & 0 \\ & & & & & & & & & & & & 1 & & & & -5 & 5 & 0 \\ & & & & & & & & & & & & & 1 & & & -2 & 3 & -1 \\ & & & & & & & & & & & & & & 1 & & -4 & 3 & 1 \\ & & & & & & & & & & & & & & & -2 & 1 & 2 \\ & & & & & & & & & & & & & & & 1 & 1 & -2 \\ & & & & & & & & & & & & & & & 1 & -3 & 1 \end{pmatrix}.$$

Then we have ${}^tUM_1U = M_0$. From this result, by observing $L = E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$, we obtain the matrix A also. \square

Remark 2.2. The above theorem says that $S(\lambda, \mu)$ has the Shioda-Inose structure for a generic point (λ, μ) (see [Mo]).

Remark 2.3. Our lattice (2.8) is isomorphic to the lattice $E_8(-1) \oplus T_{2,5,5}$. S. M. Belcastro [Be] researched 95 families of weighted projective K3 surfaces. The Picard lattices of the family No.30 and No.86 in her table are $E_8(-1) \oplus T_{2,5,5}$.

3 Period differential equation

Set $F(x, y, z) = xyz^2(x + y + z + 1) + \lambda xyz + \mu$.

The unique holomorphic 2-form on $S(\lambda, \mu)$ is given by

$$\omega = \frac{zdz \wedge dx}{\partial F / \partial y}, \quad (3.1)$$

up to a constant factor.

Theorem 3.1. When (λ, μ) is in a sufficiently small neighborhood of $(0, 0)$, the period of $S(\lambda, \mu)$ on a certain 2-cycle Γ is a holomorphic function given by the following power series.

$$\eta(\lambda, \mu) = \int_{\Gamma} \omega = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^m \frac{(5m+2n)!}{n!(m!)^3(2m+n)!} \lambda^n \mu^m. \quad (3.2)$$

Proof. When (λ, μ) is sufficiently small, $S(\lambda, \mu)$ in (1.3) is regarded as a double cover by the projection

$$p : (x, y, z) \mapsto (x, z).$$

Let $\xi_1(x, z), \xi_2(x, z)$ be the two roots of $F(x, y, z) = 0$ in y . Then we have

$$F(x, y, z) = xz^2(y - \xi_1(x, z))(y - \xi_2(x, z)).$$

and

$$\frac{\partial F(x, y, z)}{\partial y} = xz^2((y - \xi_1(x, z)) + (y - \xi_2(x, z))).$$

Therefore, at $(x, \xi_1(x, z), z) \in S(\lambda, \mu)$,

$$\frac{\partial F(x, \xi_1(x, z), z)}{\partial y} = xz^2(\xi_1(x, z) - \xi_2(x, y)).$$

We have a local inverse map of p

$$q : (x, z) \mapsto (x, \xi_1(x, z), z).$$

Let $\gamma_1(\gamma_2, \gamma_3, \text{resp.})$ be a cycle in x -plane (y -plane, z -plane, resp.) which goes around the origin once in the positive direction. We can assume that there exists $\delta > 0$ such that it holds

$$|\xi_1(x, z)| - |\xi_2(x, z)| \geq \delta$$

for any $(x, z) \in \gamma_1 \times \gamma_3$. We assume $x = -1$ stays outside of γ_1 , $z = -1 - x$ stays outside of γ_3 for any $x \in \gamma_1$, and $y = \xi_1(x, z)$ stays inside of γ_2 and $y = \xi_2(x, z)$ and $-1 - x - z$ stay outside of γ_2 for any $(x, z) \in \gamma_1 \times \gamma_3$. Moreover, by taking a neighborhood U of the origin sufficiently small, we can assume

$$|\lambda xyz + \mu| \leq |xyz^2(x + y + z + 1)|$$

for any $(x, y, z) \in \gamma_1 \times \gamma_2 \times \gamma_3$ and $(\lambda, \mu) \in U$.

Let us calculate the period integral on the 2-cycle $q(\gamma_1 \times \gamma_3)$ on $S(\lambda, \mu)$. Let ω be the holomorphic 2-form given in (3.1). By the residue theorem,

$$\begin{aligned} \iint_{q(\gamma_1 \times \gamma_3)} \omega &= \iint_{\gamma_3 \times \gamma_1} \frac{zdz \wedge dx}{xz^2(\xi_1(x, z) - \xi_2(x, z))} \\ &= \frac{1}{2\pi\sqrt{-1}} \iiint_{\gamma_3 \times \gamma_1 \times \gamma_2} \frac{zdz \wedge dx \wedge dy}{xz^2(y - \xi_1(x, z))(y - \xi_2(x, z))} \\ &= \frac{1}{2\pi\sqrt{-1}} \iiint_{\gamma_3 \times \gamma_1 \times \gamma_2} \frac{zdz \wedge dx \wedge dy}{xyz^2(x + y + z + 1) + \lambda xyz + \mu}. \end{aligned} \quad (3.3)$$

By the residue theorem and the binomial theorem, we have

$$\begin{aligned} &\frac{1}{2\pi\sqrt{-1}} \iiint_{\gamma_3 \times \gamma_1 \times \gamma_2} \frac{zdz \wedge dx \wedge dy}{xyz^2(x + y + z + 1) + \lambda xyz + \mu} \\ &= \frac{1}{2\pi\sqrt{-1}} \iiint_{\gamma_3 \times \gamma_1 \times \gamma_2} \frac{1}{xyz^2(x + y + z + 1)} \frac{zdz \wedge dx \wedge dy}{1 + \frac{\lambda xyz + \mu}{xyz^2(x + y + z + 1)}} \\ &= \frac{1}{2\pi\sqrt{-1}} \sum_{l=0}^{\infty} \iiint_{\gamma_3 \times \gamma_1 \times \gamma_2} \frac{z(-\lambda xyz - \mu)^l}{(xyz^2(x + y + z + 1))^{l+1}} dz \wedge dx \wedge dy \\ &= \frac{1}{2\pi\sqrt{-1}} \sum_{m,n=0}^{\infty} \iiint_{\gamma_3 \times \gamma_1 \times \gamma_2} \binom{m+n}{m} \frac{x^n y^n z^{n+1} dz \wedge dx \wedge dy}{(xyz^2(x + y + z + 1))^{m+n+1}} (-\lambda)^n (-\mu)^m \\ &= \frac{1}{2\pi\sqrt{-1}} \sum_{m,n=0}^{\infty} \iiint_{\gamma_3 \times \gamma_1 \times \gamma_2} \frac{(m+n)!}{n!m!} \frac{dz \wedge dx \wedge dy}{x^{m+1} y^{m+1} z^{2m+n+1} (x + y + z + 1)^{m+n+1}} (-\lambda)^n (-\mu)^m \\ &= \sum_{n,m=0}^{\infty} \iint_{\gamma_3 \times \gamma_1} \frac{(2m+n)!}{(m!)^2 n!} (-1)^m \frac{dz \wedge dx}{x^{m+1} z^{2m+n+1} (x + z + 1)^{2m+n+1}} (-\lambda)^n (-\mu)^m \\ &= (2\pi\sqrt{-1}) \sum_{n,m=0}^{\infty} \int_{\gamma_3} \frac{(3m+n)!}{(m!)^3 n!} \frac{dz}{z^{2m+n+1} (z + 1)^{3m+n+1}} (-\lambda)^n (-\mu)^m \\ &= (2\pi\sqrt{-1})^2 \sum_{n,m=0}^{\infty} (-1)^m \frac{(5m+2n)!}{(m!)^3 n! (2m+n)!} \lambda^n \mu^m. \end{aligned}$$

The above power series is holomorphic on U . □

In the following we use the notation

$$\theta_\lambda = \lambda \frac{\partial}{\partial \lambda}, \quad \theta_\mu = \mu \frac{\partial}{\partial \mu}.$$

Proposition 3.1. *Set*

$$\begin{cases} L_1 &= \theta_\lambda(\theta_\lambda + 2\theta_\mu) - \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2), \\ L_2 &= \lambda^2\theta_\mu^3 + \mu\theta_\lambda(\theta_\lambda - 1)(2\theta_\lambda + 5\theta_\mu + 1). \end{cases} \quad (3.4)$$

The period $\eta(\lambda, \mu)$ in (3.2) satisfies the system $L_1 = L_2 = 0$.

Proof. Set

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{pmatrix}, \quad \beta = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By this data, we can obtain the GKZ hypergeometric differential equation which has a solution

$$\begin{aligned} & \iiint_{\Delta} P^{-1} t_1^{-1} t_2^{-1} t_3^{-1} dt_1 \wedge dt_2 \wedge dt_3 \\ &= \iiint_{\Delta} \frac{t_3 dt_1 \wedge dt_2 \wedge dt_3}{(t_1 t_2 t_3^2 (a_1 + a_2 t_1 + a_3 t_2 + a_4 t_3) + a_5 t_1 t_2 t_3 + a_6)}, \end{aligned} \quad (3.5)$$

where

$$P = a_1 + a_2 t_1 + a_3 t_2 + a_4 t_3 + a_5 \frac{1}{t_3} + a_6 \frac{1}{t_1 t_2 t_3^2},$$

and Δ is a twisted cycle. By the parameter transformation (1.2), (3.5) is transformed to

$$\frac{1}{a_1} \iiint_{\Delta} \frac{z dx \wedge dy \wedge dz}{xyz^2(x + y + z + 1) + \lambda xyz + \mu} = \frac{1}{a_1} \eta(\lambda, \mu).$$

Set $\theta_j = a_j \frac{\partial}{\partial a_j}$. The above mentioned system of GKZ is given by the following equations (3.6), (3.7) and (3.8) :

$$\begin{cases} (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6)\eta = -\eta, \\ (\theta_2 - \theta_6)\eta = 0, \\ (\theta_3 - \theta_6)\eta = 0, \\ (\theta_4 - \theta_5 - 2\theta_6)\eta = \eta, \end{cases} \quad (3.6)$$

$$\frac{\partial^2}{\partial a_4 \partial a_5} \eta = \frac{\partial^2}{\partial a_1^2} \eta, \quad (3.7)$$

$$\frac{\partial^3}{\partial a_2 \partial a_3 \partial a_6} \eta = \frac{\partial^3}{\partial a_1 \partial a_5^2} \eta. \quad (3.8)$$

By (1.2), we have

$$\theta_\lambda = \theta_5, \quad \theta_\mu = \theta_6.$$

So, from (3.6) we have

$$\begin{cases} \theta_2 \eta = \theta_\mu \eta, \\ \theta_3 \eta = \theta_\mu \eta, \\ \theta_4 \eta = (\theta_\lambda + 2\theta_\mu) \eta, \\ \theta_1 \eta = (-2\theta_\lambda - 5\theta_\mu - 1) \eta. \end{cases}$$

From (3.7), we have

$$\left\{ \begin{array}{l} \frac{\partial^2}{\partial a_4 \partial a_5} \eta = \frac{1}{a_4 a_5} \theta_4 \theta_5 \eta = \frac{1}{a_4 a_5} (\theta_\lambda + 2\theta_\mu) \theta_\lambda \eta, \\ \frac{\partial^3}{\partial a_1^2} \eta = \frac{1}{a_1^2} \theta_1 (\theta_1 - 1) \eta = \frac{1}{a_1^2} (2\theta_\lambda + 5\theta_\mu + 1) (2\theta_\lambda + 5\theta_\mu + 2) \eta. \end{array} \right.$$

Hence we obtain

$$(\theta_\lambda + 2\theta_\mu) \eta = \lambda (2\theta_\lambda + 5\theta_\mu + 1) (2\theta_\lambda + 5\theta_\mu + 2) \eta.$$

Similarly, from (3.8) we have

$$\left\{ \begin{array}{l} \frac{\partial^3}{\partial a_2 \partial a_3 \partial a_6} \eta = \frac{1}{a_2 a_3 a_6} \theta_2 \theta_3 \theta_6 \eta = \frac{1}{a_2 a_3 a_6} \theta_\mu^3 \eta, \\ \frac{\partial^3}{\partial a_1 \partial a_5^2} \eta = \frac{1}{a_5^2} \theta_1 \theta_5 (\theta_5 - 1) \eta = \frac{1}{a_1 a_5^2} (-2\theta_\lambda - 5\theta_\mu - 1) \theta_\lambda (\theta_\lambda - 1) \eta, \end{array} \right.$$

hence

$$\lambda^2 \theta_\mu^3 \eta = -\mu (2\theta_\lambda + 5\theta_\mu + 1) \theta_\lambda (\theta_\lambda - 1) \eta.$$

□

We can obtain a 6×6 Pfaffian from $L_1 = L_2 = 0$. This system is integrable. So, the system $L_1 = L_2 = 0$ has a 6-dimensional solution space. However, as we remarked in Corollary 2.2, we expect a system of differential equations with 4-dimensional solution space. It suggests that the system $L_1 = L_2 = 0$ is reducible.

Theorem 3.2. (1) *Set*

$$\left\{ \begin{array}{l} L_1 = \theta_\lambda (\theta_\lambda + 2\theta_\mu) - \lambda (2\theta_\lambda + 5\theta_\mu + 1) (2\theta_\lambda + 5\theta_\mu + 2), \\ L_3 = \lambda^2 (4\theta_\lambda^2 - 2\theta_\lambda \theta_\mu + 5\theta_\mu^2) - 8\lambda^3 (1 + 3\theta_\lambda + 5\theta_\mu + 2\theta_\lambda^2 + 5\theta_\lambda \theta_\mu) + 25\mu \theta_\lambda (\theta_\lambda - 1). \end{array} \right. \quad (3.9)$$

The period η in (3.2) satisfies $L_1 = L_3 = 0$.

(2) *The solution space of the system $L_1 = L_3 = 0$ is 4-dimensional.*

Proof. (1) We determine L_3 by the method of indeterminate coefficients. Set $D = f_1 + f_2 \theta_\lambda + f_3 \theta_\mu + f_4 \theta_\lambda^2 + f_5 \theta_\lambda \theta_\mu + f_6 \theta_\mu^2$, where $f_1 \cdots f_6 \in \mathbb{C}[\lambda, \mu]$. We can determine the polynomials f_1, \dots, f_6 so that D satisfies $D\eta = 0$ (η is given by (3.2)) and is independent of L_1 . Then we obtain the above L_3 .

(2) By making up the Pfaffian system of $L_1 = L_3 = 0$, we can show the required statement. Set $\varphi = {}^t(1, \theta_\lambda, \theta_\mu, \theta_\lambda^2)$. We obtain the Pfaffian system $\Omega = A_4 d\lambda + B_4 d\mu$ with $d\varphi = \Omega\varphi$ by the following way. Setting

$$\left\{ \begin{array}{l} t_4 = \lambda^2 (4\lambda - 1)^3 - 2(2 + 25\lambda(20\lambda - 1))\mu - 3125\mu^2, \\ s_4 = 1 - 15\lambda - 100\lambda^2, \end{array} \right.$$

we have

$$A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{11}/s_4 & a_{12}/(2\lambda s_4) & a_{13}/(2s_4) & a_{14}/(2\lambda s_4) \\ a_{21}/(s_4 t_4) & a_{22}/(2s_4 t_4) & a_{23}/(2s_4 t_4) & a_{24}/(2s_4 t_4) \end{pmatrix},$$

with

$$\begin{cases} a_{11} = \lambda(1 + 20\lambda), \\ a_{12} = 6\lambda^2 + 120\lambda^3 + 125\mu, \\ a_{13} = 5\lambda(3 + 40\lambda), \\ a_{14} = -(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu), \\ a_{21} = -\lambda^3(2 + 2125\mu + \lambda(-17 + 616\lambda - 2320\lambda^2 + 2500(9 + 80\lambda)\mu)), \\ a_{22} = -(-2\lambda^3(-1 + 4\lambda)(8 + 5\lambda(-13 + 4\lambda(83 + 40\lambda))) \\ \quad + (-16 + 5\lambda(94 + 5\lambda(59 + 10\lambda(-73 + 20\lambda(37 + 160\lambda))))))\mu + 3125(-4 + 5\lambda(21 + 200\lambda))\mu^2), \\ a_{23} = -\lambda^3(22 + 26875\mu + \lambda(-47 + 300000\mu + 100\lambda(51 + 4\lambda(-49 + 20\lambda) + 20000\mu))), \\ a_{24} = 12r_4s_4 + 3r_4(15\lambda - 2) + 2s_4(-3(1 - 4\lambda)^2\lambda^2(-1 + 10\lambda) + 75\lambda(-1 + 40\lambda)\mu), \end{cases}$$

and

$$B_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ b_{11}/s_4 & b_{12}/(2\lambda s_4) & b_{13}/(2s_4) & b_{14}/(2\lambda s_4) \\ b_{21}/(s_4) & b_{22}/(\lambda^2 s_4) & b_{23}/(s_4) & b_{24}/(\lambda^2 s_4) \\ b_{31}/(t_4 s_4) & b_{32}/(2\lambda t_4 s_4) & b_{33}/(2t_4 s_4) & b_{34}/(2\lambda t_4 s_4) \end{pmatrix},$$

with

$$\begin{cases} b_{11} = \lambda(1 + 20\lambda), \\ b_{12} = 6\lambda^2 + 120\lambda^3 + 125\mu, \\ b_{13} = 5\lambda(3 + 40\lambda), \\ b_{14} = -(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu), \\ b_{21} = -2\lambda(-1 + 4\lambda), \\ b_{22} = -(6\lambda^3(-1 + 4\lambda) - 5\mu + 50\lambda\mu), \\ b_{23} = -\lambda(-11 + 20\lambda), \\ b_{24} = -((1 - 4\lambda)^2\lambda^2 - (5 - 50\lambda)\mu), \\ b_{31} = -(4(1 - 4\lambda)^2\lambda^4(7 + 20\lambda) \\ \quad - \lambda(-4 + 25\lambda(-3 + 2\lambda(-7 + 20\lambda(1 + 80\lambda))))\mu + 3125\lambda(1 + 20\lambda)\mu^2), \\ b_{32} = -(24(1 - 4\lambda)^2\lambda^5(7 + 20\lambda) - 2\lambda(-4 + 5\lambda(8 + \lambda(-43 + 10\lambda(-57 + 20\lambda(7 + 160\lambda))))))\mu \\ \quad - 125(-4 + 25\lambda(-3 + 32\lambda(1 + 10\lambda)))\mu^2 + 390625\mu^3), \\ b_{33} = -(4\lambda^3(-1 + 4\lambda)(-1 + 2\lambda(-32 + 25\lambda(1 + 12\lambda))) + 15625\lambda(3 + 40\lambda)\mu^2 \\ \quad - 5\lambda(-12 + 5\lambda(-1 + 10\lambda)(33 + 20\lambda(23 + 160\lambda)))\mu), \\ b_{34} = -(4\lambda^4(-1 + 4\lambda)^3(7 + 20\lambda) + 3\lambda(-4 + \lambda(31 - 490\lambda + 76000\lambda^3))\mu \\ \quad + 250(-2 + 25\lambda(-2 + \lambda(11 + 260\lambda)))\mu^2 - 390625\mu^3). \end{cases}$$

For our Pfaffian system Ω we see the integrable condition $d\Omega = \Omega \wedge \Omega$. Therefore the system $L_1 = L_3 = 0$ has the 4-dimensional solution space. \square

Remark 3.1. *N. Takayama and M. Nakayama [TN] reported that they obtain the rank of differential equations for Fano polytopes with at most 6 vertices by a specially adapted use of D-module algorithm.*

By changing the system $\varphi = {}^t(1, \theta_\lambda, \theta_\mu, \theta_\lambda^2)$ to other ones, we see $s_4 = 0$ is not a singularity. Together with the singularity of $\theta_\lambda, \theta_\mu$ we obtain the singular locus of the system $L_1 = L_3 = 0$:

$$\lambda = 0, \quad \mu = 0, \quad \lambda^2(4\lambda - 1)^3 - 2(2 + 25\lambda(20\lambda - 1))\mu - 3125\mu^2 = 0. \quad (3.10)$$

This is the locus mentioned in Remark 1.1.

4 Monodromy

Take a generic point $(\lambda_0, \mu_0) \in \Lambda$. Let $S_0 = S(\lambda_0, \mu_0)$ be a reference surface. Set $L = H_2(S_0, \mathbb{Z})$. $L' = \text{NS}(S_0)$ is generated by the system (2.6). Recalling the argument of Section 2, we have a \mathbb{Z} -basis $\{\gamma_1, \dots, \gamma_{22}\}$ of L with $\langle \gamma_5, \dots, \gamma_{22} \rangle_{\mathbb{Z}} = L'$.

Let A be the intersection matrix of transcendental larrice displayed in (2.9). Set

$$PO(A, \mathbb{Z}) = \{P \in GL(4, \mathbb{Z}) \mid {}^t P A P = A\}. \quad (4.1)$$

It acts on \mathcal{D} by

$${}^t \xi \mapsto P^t \xi \quad (\xi \in \mathcal{D}, P \in PO(A, \mathbb{Z})).$$

Recall that \mathcal{D} is composed of two connected components:

$$\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-.$$

Definition 4.1. Let $PO^+(A, \mathbb{Z})$ denote the subgroup of $PO(A, \mathbb{Z})$ given by

$$\{g \in PO(A, \mathbb{Z}) \mid g(\mathcal{D}_\pm) = \mathcal{D}_\pm\}.$$

Remark 4.1. $PO(A, \mathbb{Z})$ is generated by the system:

$$\left\{ \begin{array}{l} G_1 = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, G_2 = \begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, G_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \\ H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{array} \right.$$

G_1, G_2, G_3, H_2 generate $PO^+(A, \mathbb{Z})$ (see [I1] or [Ma]).

In this section we show that the projective monodromy group of our period map is isomorphic to the orthogonal group $PO^+(A, \mathbb{Z})$. To prove this we apply the Torelli type theorem for polarized $K3$ surfaces.

4.1 The Torelli theorem for P-marked $K3$ surfaces

First, we state necessary properties of polarized $K3$ surfaces.

Definition 4.2. Let S be an algebraic $K3$ surface. An isomorphism $\psi : H_2(S, \mathbb{Z}) \rightarrow L = H_2(S_0, \mathbb{Z})$ is said to be a P -marking in case we have :

- (i) $\psi^{-1}(L') \subset \text{NS}(S)$,
- (ii) $\psi^{-1}(F), \psi^{-1}(O), \psi^{-1}(P), \psi^{-1}(Q), \psi^{-1}(b_j), \psi^{-1}(b'_j) (1 \leq j \leq 7)$ are all effective divisors,
- (iii) $\psi^{-1}(F)$ is nef.

A pair (S, ψ) of a $K3$ surface and a P -marking is called a P -marked $K3$ surface.

A S -marked $K3$ surface $(S(\lambda, \mu), \psi)$ is a P -marked $K3$ surface.

Definition 4.3. Two P -marked $K3$ surfaces (S_1, ψ_1) and (S_2, ψ_2) are said to be isomorphic if we have a biholomorphic map $f : S_1 \rightarrow S_2$ with

$$\psi_2 \circ f_* \circ \psi_1^{-1} = id_L.$$

Two P -marked $K3$ surfaces (S_1, ψ_1) and (S_2, ψ_2) are said to be equivalent if we have a biholomorphic map $f : S_1 \rightarrow S_2$ with

$$\psi_2 \circ f_* \circ \psi_1^{-1}|_{L'} = id_{L'}.$$

The period of a P -marked $K3$ surface (S, ψ) is defined by

$$\Phi(S, \psi) = \left(\int_{\psi^{-1}(\gamma_1)} \omega : \cdots : \int_{\psi^{-1}(\gamma_4)} \omega \right). \quad (4.2)$$

We use some general facts exposed in [KSTT] .

Proposition 4.1. (Pjateckiĭ-Šapiro and Šafarevič [PS]) *Let S be a K3 surface, then we have the following:*

- (1) *Suppose $D \in \text{NS}(S)$ satisfies $(D \cdot D) = 0, D \neq 0$. Then there exists an isometry γ of $\text{NS}(S)$ such that $\gamma(D)$ becomes to be effective and nef.*
- (2) *Suppose $D \in \text{NS}(S)$ is effective, nef and $(D \cdot D) = 0$. Then, for certain $m \in \mathbb{N}$ and an elliptic curve $E \in S$, $D = m[E]$.*
- (3) *A linear system of an elliptic curve E on S determines an elliptic fibration $S \rightarrow \mathbb{P}^1(\mathbb{C})$.*

Proposition 4.2. *A P -marked K3 surface (S, ψ) is realized as an elliptic K3 surface which has $\psi^{-1}(F)$ as a general fibre. Especially, if S is realized as a K3 surface $S(\lambda, \mu)$ by the Kodaira normal form for some $(\lambda, \mu) \in \Lambda$, it is a S -marked K3 surface.*

Proof. Set $D = \psi^{-1}(F) \in \text{Div}(S)$. By Definition 4.3, D is effective, nef, and $(D \cdot D) = 0$. According to Proposition 4.1 (2), there exists a positive integer m and an elliptic curve E such that $D = m[E]$. However

$$m(E \cdot \psi^{-1}(O)) = (D \cdot \psi^{-1}(O)) = (F \cdot O) = 1.$$

Therefore we have $m = 1$. Proposition 4.1 (3) says that there is an elliptic fibration $\pi : S \rightarrow \mathbb{P}^1(\mathbb{C})$ which has $D = \psi^{-1}(F)$ as a general fibre. \square

Let X be the isomorphic classes of P -marked K3 surfaces and set

$$[X] = X / \text{P-marked equivalence}.$$

By (4.2), we obtain our period map $\Phi : X \rightarrow \mathbb{P}^3(\mathbb{C})$.

We state the Torelli type theorem for polarized K3 surfaces.

Theorem 4.1. *We have the following properties.*

- (1) $\Phi(X) \subset \mathcal{D}$.
- (2) $\Phi : X \rightarrow \mathcal{D}$ is a bijective correspondence.
- (3) *Let S and S' be algebraic K3 surfaces. Suppose an isometry $\varphi : H_2(S, \mathbb{Z}) \rightarrow H_2(S', \mathbb{Z})$ preserves ample classes. Then there exists a biholomorphic map $f : S \rightarrow S'$ such that $\varphi = f_*$.*

Here we prove the following two key lemmas.

Lemma 4.1. *A P -marked K3 surface (S, ψ) is equivalent to the P -marked reference surface (S_0, ψ_0) if and only if $\Phi(S, \psi) = g \circ \Phi(S_0, \psi_0)$ for some $g \in PO(A, \mathbb{Z})$.*

Proof. The necessity is obvious. We prove the sufficiency. Suppose $\Phi((S_0, \psi_0)) = p \in \mathcal{D}$, and take $g \in PO(A, \mathbb{Z})$. According to Theorem 4.1 (2), we can take a P -marked K3 surface (S_g, ψ_g) such that $\Phi(S_g, \psi_g) = g \circ \Phi(S_0, \psi_0)$. Let L'' be the transcendental lattice given in (2.2). Note $g \in \text{Aut}(L'') = PO(A, \mathbb{Z})$. Due to Nikulin [N], $g : L'' \rightarrow L''$ can be extended to an isomorphism $\tilde{g} : L \rightarrow L$ which satisfies $\tilde{g}|_{L'} = \text{id}|_{L'}$. Then, by Theorem 4.1 (3), there is a biholomorphic map $f : S_0 \rightarrow S_g$ such that $f_* = \tilde{g}$. Therefore two P -marked K3 surfaces (S_0, ψ_0) and (S_g, ψ_g) are equivalent. \square

Remark 4.2. $PO(A, \mathbb{Z})$ is a reflection group (see [Ma]).

According to the Torelli theorem and Lemma 4.1, we can identify $[X]$ with $\mathcal{D}/PO(A, \mathbb{Z})$.

Lemma 4.2. *Let (S, ψ) be a P -marked K3 surface which is equivalent to (S_0, ψ_0) . Then (S, ψ) has a unique canonical elliptic fibration $(S, \pi, \mathbb{P}^1(\mathbb{C}))$ that is given by the Kodaira normal form of $S(\lambda_0, \mu_0)$ not coming from any other $(\lambda, \mu) \in \Lambda$.*

Proof. From Proposition 4.2, (S, ψ) ((S_0, ψ_0) , resp.) has an elliptic fibration $(S, \pi, \mathbb{P}^1(\mathbb{C}))$ ($(S_0, \pi_0, \mathbb{P}^1(\mathbb{C}))$, resp.) with a general fibre $\psi^{-1}(F)$ ($\psi_0^{-1}(F)$, resp.). Because (S, ψ) and (S_0, ψ_0) are equivalent as P-marked K3 surfaces, we have a biholomorphic map $f : S \rightarrow S_0$ such that

$$\psi_0 \circ f_* = \psi \quad (f_* : H_2(S, \mathbb{Z}) \simeq H_2(S_0, \mathbb{Z})).$$

Through the identification $L = H_2(S_0, \mathbb{Z})$, we have

$$f_* = \psi.$$

It means that f preserves general fibres of S and S_0 . According to the uniqueness of the fibration (Lemma 2.1), $(S, \pi, \mathbb{P}^1(\mathbb{C}))$ and $(S_0, \pi_0, \mathbb{P}^1(\mathbb{C}))$ are isomorphic as elliptic surfaces. According to Proposition 2.1, we have $\varphi \in \text{Aut}(\mathbb{P}^1(\mathbb{C}))$ such that $\varphi \circ \pi = \pi_0 \circ f$.

Let $y^2 = 4x^3 - g_2(z)x - g_3(z)$ ($y^2 = 4x^3 - g_{20}(z)x - g_{30}(z)$, resp.) be the Kodaira normal form of $(S, \pi, \mathbb{P}^1(\mathbb{C}))$ ($(S_0, \pi_0, \mathbb{P}^1(\mathbb{C}))$, resp.). According to Proposition 1.6, we can assume $\pi^{-1}(0) = I_3$ and $\pi^{-1}(\infty) = I_{15}$. So as in the proof of Lemma 1.1, φ is given by $z \mapsto az, a \in \mathbb{C} - 0$. Let j (j_0 , resp.) be the j -invariant of S (S_0 , resp.) and let D (D_0 , resp.) be the discriminant of S (S_0 , resp.). By Proposition 1.6, we know that $D = D_0 \circ \varphi$ and $j = j_0 \circ \varphi$. Observing the expressions (1.5), (1.6) around $z = \infty$ and the definition of j -function (1.9), it is necessary that $a^3 = 1$. By the transformation $z \mapsto \omega z$ or $z \mapsto \bar{\omega} z$ (where ω is a cubic root of unity) we can assume $a = 1$. Comparing j with j_0 and D with D_0 we have $g_2^3 = g_{20}^3$ and $g_3^3 = g_{30}^3$. However by the transformations in the form $x \mapsto \omega x$ or $x \mapsto \bar{\omega} x$ or $y \mapsto -y$, we can assume $g_2 = g_{20}$ and $g_3 = g_{30}$. Hence, as in the proof of Lemma 1.1, we have the required statement. \square

Remark 4.3. According to the above two lemmas, Λ is embedded in $[X]$.

4.2 Projective monodromy group

Theorem 4.2. The projective monodromy group is isomorphic to $PO^+(A, \mathbb{Z})$.

Proof. Let $* = (\lambda_0, \mu_0)$ be a generic point of Λ . Set $S_0 = S(\lambda_0, \mu_0)$. We have $\text{NS}(S_0) \simeq L'$. Note that every such point $*$ in the affine part is contained in Λ . Let G be the projective monodromy group induced from $\pi_1(\Lambda, *)$. The inclusion $G \subset PO^+(A, \mathbb{Z})$ is apparent.

So we prove the opposite inclusion $PO^+(A, \mathbb{Z}) \subset G$. Take an element $g \in PO^+(A, \mathbb{Z})$, and let $p = \Phi(S_0, \psi_0) \in \mathcal{D}$ and let $q = g(p) \in \mathcal{D}$. p, q are in the same connected component of \mathcal{D} . So we can assume that $p, q \in \mathcal{D}^+$. Let α be an arc connecting p and q in \mathcal{D}^+ . By the Torelli theorem, we have $[\Phi^{-1}(\alpha)] \subset [X]$. By Lemma 5.1 and Lemma 5.2, we have $q = \Phi(S_0, \psi)$ so that (S_0, ψ) is equivalent to (S_0, ψ_0) . Hence the end point of $[\Phi^{-1}(\alpha)]$ is (λ_0, μ_0) .

Next, we show that we can choose α so that $[\Phi^{-1}(\alpha)] \subset \Lambda$. For this purpose, it is enough to show that Λ is a Zariski open set in some compactification K of $[X]$. Here we note that the compact (λ, μ) space $\mathbb{P}^2(\mathbb{C})$ and K are birationally equivalent and they contain Λ as a common open set. Λ is Zariski open set in $\mathbb{P}^2(\mathbb{C})$. So it is Zariski open in K , also. Hence we obtained the required inclusion. \square

Remark 4.4. This result is essentially found in the research [I2] of T. Ishige on a family of K3 surfaces coming from another reflexive polytope. He discovered this result by a precise computer approximation of the generator system of the monodromy group. However it is not given an exact error estimation there. So we gave here an independent proof for our case based on the Torelli type theorem for K3 surfaces.

5 Period differential equation and the Hilbert modular orbifold for the field $\mathbb{Q}(\sqrt{5})$

Let \mathcal{O} be the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{5})$. Set $\mathbb{H}_\pm = \{z \in \mathbb{C} \mid \pm z > 0\}$. The Hilbert modular group $PSL(2, \mathcal{O})$ acts on $(\mathbb{H}_+ \times \mathbb{H}_+) \cup (\mathbb{H}_- \times \mathbb{H}_-)$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (z_1, z_2) \mapsto \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right),$$

for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2, \mathcal{O})$, where $'$ means the conjugate in k .

According to the method introduced in [I1], we have the following facts. Set

$$W = \begin{pmatrix} 1 & 1 \\ (1 - \sqrt{5})/2 & (1 + \sqrt{5})/2 \end{pmatrix}.$$

It holds

$$A = U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = U \oplus WU^tW.$$

The correspondence

$$(z_1, z_2) \rightarrow (I_2 \oplus {}^tW^{-1}) \begin{pmatrix} z_1 z_2 \\ -1 \\ z_1 \\ z_2 \end{pmatrix}$$

defines a biholomorphic isomorphism

$$\iota : (\mathbb{H}_+ \times \mathbb{H}_+) \cup (\mathbb{H}_- \times \mathbb{H}_-) \rightarrow \mathcal{D}.$$

Setting

$$\begin{cases} \tau : (z_1, z_2) \rightarrow (z_2, z_1), \\ \tau' : (z_1, z_2) \rightarrow \left(\frac{1}{z_1}, \frac{1}{z_2} \right), \end{cases}$$

we set

$$\rho(g) = \iota \circ g \circ \iota^{-1}$$

for $g \in \langle PSL(2, \mathcal{O}), \tau, \tau' \rangle$. We have $\rho(\langle PSL(2, \mathcal{O}), \tau, \tau' \rangle) = PO(A, \mathbb{Z})$. Put $\mathbb{H} = \mathbb{H}_+$. The pair (ι, ρ) gives a modular isomorphism

$$(\mathbb{H} \times \mathbb{H}, \langle PSL(2, \mathcal{O}), \tau \rangle) \simeq (\mathcal{D}_+, PO^+(A, \mathbb{Z})). \quad (5.1)$$

On the other hand there are several researches into the Hilbert modular orbifolds for the field $\mathbb{Q}(\sqrt{5})$. F. Hirzebruch [H] studied the orbifold $(\mathbb{H} \times \mathbb{H})/\langle \Gamma, \tau \rangle$ (the group Γ is given in (5.4)). In this research he used Klein's icosahedral polynomials. R. Kobayashi, K. Kushibiki and I. Naruki [KKN] studied the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ and determined its branch divisor. T. Sato [Sa] studied the uniformizing differential equation (see Definition 5.3) of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$.

Because of the modular isomorphism (5.1) and Theorem 4.2, we expect that our period differential equation (3.9) is related to the uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$.

In this section we realize the explicit relation between our period differential equation and the uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$. We discover the exact birational transformation (5.11) from our (λ, μ) -space to (x, y) -space, where (x, y) are affine coordinates expressed by Klein's icosahedral polynomials in (5.5). Moreover we show that the uniformizing differential equation with the normalization factor (5.15) is equal to our period differential equation (3.9).

5.1 Linear differential equations in 2 variables of rank 4

First we survey the study of T. Sasaki and M. Yoshida [SY]. It supplies a fundamental tool for the research into uniformizing differential equations of the Hilbert modular orbifolds.

We consider a system of linear differential equations

$$\begin{cases} Z_{XX} = LZ_{XY} + AZ_X + BZ_Y + PZ, \\ Z_{YY} = MZ_{XY} + CZ_X + DZ_Y + QZ, \end{cases} \quad (5.2)$$

where (X, Y) are independent variables and Z is the unknown. We assume its solution space is 4-dimensional.

Definition 5.1. *We call the symmetric 2-tensor*

$$L(dX)^2 + 2(dX)(dY) + M(dY)^2 \quad (5.3)$$

the holomorphic conformal structure of (5.2) .

Remark 5.1. *The above symmetric 2-tensor (5.3) is equal to the holomorphic conformal structure of the complex surface patch embedded in $\mathbb{P}^3(\mathbb{C})$ defined by the projective solution of (5.2).*

Definition 5.2. *Let Z_0, Z_1, Z_2 and Z_3 be linearly independent solutions of (5.2). Put $Z = {}^t(Z_0, Z_1, Z_2, Z_3)$. The function*

$$e^{2\theta} = \det(Z, Z_X, Z_Y, Z_{XY})$$

is called the normalization factor of (5.2).

Proposition 5.1. ([SY] (see also [Sa], p.181)) *The surface patch by the projective solution of (5.2) is a part of non degenerate quadratic surface in $\mathbb{P}^3(\mathbb{C})$ if and only if*

$$\begin{cases} A = \frac{\partial}{\partial X} \left(\frac{1}{4}\xi + \theta \right) - \frac{L}{2} \frac{\partial}{\partial Y} \left(\log(L) - \frac{1}{4}\xi + \theta \right), \\ B = \frac{L}{2} \frac{\partial}{\partial X} \left(\log(L) - \frac{3}{4}\xi - \theta \right), \\ C = \frac{M}{2} \frac{\partial}{\partial Y} \left(\log(M) - \frac{3}{4}\xi - \theta \right), \\ D = \frac{\partial}{\partial Y} \left(\frac{1}{4}\xi + \theta \right) - \frac{M}{2} \frac{\partial}{\partial X} \left(\log(M) - \frac{1}{4}\xi + \theta \right), \end{cases}$$

where $\xi = \log(1 - LM)$.

Proposition 5.2. ([SY], p.76) *Perform a coordinate change of the equation (5.2) from (X, Y) to (U, V) and denote the coefficients of the transformed equation by the same letter with bars. Then*

$$\begin{cases} \bar{L} = -\lambda/\nu, & \bar{M} = -\mu/\nu, \\ \bar{A} = (R(U)\beta - S(U)\alpha)/\nu, & \bar{B} = (R(V)\beta - S(V)\alpha)/\nu, \\ \bar{C} = (S(U)\gamma - R(U)\delta)/\nu, & \bar{D} = (S(V)\gamma - R(V)\delta)/\nu, \\ \bar{P} = (\alpha Q - \beta P)/\nu, & \bar{Q} = (\delta P - \gamma Q)/\nu, \end{cases}$$

where

$$\begin{cases} \Delta = U_X V_Y - U_Y V_X, \\ \lambda = L V_Y^2 - 2 V_X V_Y + M V_X^2, \\ \mu = L U_Y^2 - 2 U_X U_Y + M U_X^2, \\ \nu = L U_Y V_Y - U_X V_Y - U_Y V_X + M U_X V_X, \end{cases}$$

and

$$\begin{cases} \alpha = (V_X^2 - LV_X V_Y)/\Delta, & \beta = (V_Y^2 - MV_X V_Y)/\Delta, \\ \gamma = (U_X^2 - LU_X U_Y)/\Delta, & \delta = (U_Y^2 - MU_X U_Y)/\Delta, \\ R(u) = U_{XX} - (LU_{XY} + AU_X + BU_Y), \\ S(U) = U_{YY} - (MU_{XY} + CU_X + DU_Y), \\ R(V) = V_{XX} - (LV_{XY} + AV_X + BV_Y), \\ S(V) = V_{YY} - (MV_{XY} + CV_X + DV_Y). \end{cases}$$

5.2 Uniformizing differential equation of orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$

Let us sum up the facts about the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ and the result of T. Sato [Sa] on the uniformizing differential equation.

The quotient space $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ carries the structure of an orbifold.

Set

$$\Gamma = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2, \mathcal{O}) \mid \alpha \equiv \delta \equiv 1, \beta \equiv \gamma \equiv 0 \pmod{\sqrt{5}} \right\}. \quad (5.4)$$

Γ is a normal subgroup of $PSL(2, \mathcal{O})$. The quotient group $PSL(2, \mathcal{O})/\Gamma$ is isomorphic to alternating group A_5 of degree 5. A_5 is isomorphic to the icosahedral group I . Let \overline{M} be a compactification of an orbifold M . F. Hirzebruch [H] showed that $\overline{\mathbb{H} \times \mathbb{H}}/\langle \Gamma, \tau \rangle$ is isomorphic to $\mathbb{P}^2(\mathbb{C})$. Therefore $\mathbb{P}^2(\mathbb{C})$ admits an action of the alternating group A_5 . This action is equal to the action of the icosahedral group I on $\mathbb{P}^2(\mathbb{C})$ introduced by F. Klein. We list Klein's I -invariant polynomials on $\mathbb{P}^2(\mathbb{C}) = \{(A_0 : A_1 : A_2)\}$:

$$\begin{cases} A(A_0 : A_1 : A_2) = A_0^2 + A_1 A_2, \\ B(A_0 : A_1 : A_2) = 8A_0^4 A_1 A_2 - 2A_0^2 A_1^2 A_2^2 + A_1^3 A_2^3 - A_0(A_1^5 + A_2^5), \\ C(A_0 : A_1 : A_2) = 320A_0^6 A_1^2 A_2^2 - 160A_0^4 A_1^3 A_2^3 + 20A_0^2 A_1^4 A_2^4 + 6A_1^5 A_2^5 \\ \quad - 4A_0(A_1^5 + A_2^5)(32A_0^4 - 20A_0^2 A_1 A_2 + 5A_1^2 A_2^2) + A_1^{10} + A_2^{10}, \\ 12D(A_0 : A_1 : A_2) = (A_1^5 - A_2^5)(-1024A_0^{10} + 3840A_0^8 A_1 A_2 - 3840A_0^6 A_1^2 A_2^2 \\ \quad + 1200A_0^4 A_1^3 A_2^3 - 100A_0^2 A_1^4 A_2^4 + A_1^5 A_2^5) \\ \quad + A_0(A_1^{10} - A_2^{10})(352A_0^4 - 160A_0^2 A_1 A_2 + 10A_1^2 A_2^2) + (A_1^{15} - A_2^{15}). \end{cases}$$

We have the following relation:

$$144D^2 = -1728B^5 + 720ACB^3 - 80A^2C^2B + 64A^3(5B^2 - AC)^2 + C^3.$$

R. Kobayashi, K. Kushibiki and I. Naruki [KKN] showed that the compactification $\overline{(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle}$ is isomorphic to $\mathbb{P}^2(\mathbb{C})$. Let

$$\varphi : \mathbb{P}^2(\mathbb{C}) = \overline{(\mathbb{H} \times \mathbb{H})/\langle \Gamma, \tau \rangle} \rightarrow \overline{(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle} = \mathbb{P}^2(\mathbb{C})$$

be a rational map defined by

$$(A_0 : A_1 : A_2) \mapsto (A^5 : A^2 B : C).$$

φ is a holomorphic map of $\mathbb{P}^2(\mathbb{C}) - \{A = 0\}$ to $\mathbb{P}^2(\mathbb{C}) - (\text{a line at infinity } L_\infty) \subset \overline{(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle}$.

Set

$$x = \frac{B}{A^3}, \quad y = \frac{C}{A^5}. \quad (5.5)$$

x and y are the affine coordinates identifying $(1 : x : y) \in \mathbb{P}^2(\mathbb{C}) - L_\infty$ with $(x, y) \in \mathbb{C}^2$.

Proposition 5.3. ([KKN]) *The branch locus of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ in $\mathbb{P}^2(\mathbb{C}) - L_\infty = \mathbb{C}^2$ is, using the affine coordinates (5.5),*

$$D_0 = y(1728x^5 - 720x^3y + 80xy^2 - 64(5x^2 - y)^2 - y^3) = 0$$

of index 2. The orbifold structure on $\overline{(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle}$ is given by $(\mathbb{P}^2(\mathbb{C}), 2D + \infty L_\infty)$.

We note that $\mathbb{H} \times \mathbb{H}$ is embedded in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is isomorphic to a non-degenerate quadric surface in $\mathbb{P}^3(\mathbb{C})$. Let $\pi : \mathbb{H} \times \mathbb{H} \rightarrow (\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ be the canonical projection. The multivalued inverse map π^{-1} is called the developing map of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$.

Definition 5.3. *Let us consider a system of linear differential equations on the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ with 4-dimensional solution space. Let z_0, z_1, z_2, z_3 be linearly independent solutions of the system. If*

$$M \rightarrow \mathbb{P}^3(\mathbb{C}) : p \mapsto (z_0(p) : z_1(p) : z_2(p) : z_3(p))$$

gives the developing map on the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$, we call this system the uniformizing differential equation of the orbifold.

From Proposition 5.3 T.Sato obtained the following result.

Theorem 5.1. ([Sa], Example. 4) *The holomorphic conformal structure of the uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ is*

$$\frac{-20(4x^2 + 3xy - 4y)}{36x^2 - 32x - y}(dx)^2 + 2(dx)(dy) + \frac{-2(54x^3 - 50x^2 - 3xy + 2y)}{5y(36x^2 - 32x - y)}(dy)^2, \quad (5.6)$$

where (x, y) is the affine coordinates in (5.5) .

Let

$$\begin{cases} z_{xx} = Lz_{xy} + Az_x + Bz_y + Pz, \\ z_{yy} = Mz_{xy} + Cz_x + Dz_y + Qz \end{cases} \quad (5.7)$$

be the uniformizing differential equation of $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$, where (x, y) is the affine coordinates in (5.5) . We have already obtained the coefficients L and M (see Definition 5.1 and Theorem 5.1). If the normalization factor of (5.7) is given, the coefficients A, B, C and D are determined by Proposition 5.1. The other coefficients P and Q are determined by the integrability condition of (5.7).

Remark 5.2. T.Sato [Sa] determined the uniformizing differential equation of $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ with the normalization factor

$$e^{2\theta} = \frac{-36x^2 + 32x + y}{y^{1/2}(1728x^5 - 720x^3y + 80xy^2 - 64(5x^2 - y)^2 - y^3)^{3/2}}. \quad (5.8)$$

That is

$$\begin{cases} z_{xx} = Lz_{xy} + A_s z_x + B_s z_y + P_s z, \\ z_{yy} = Mz_{xy} + C_s z_x + D_s z_y + Q_s z \end{cases}$$

with

$$\begin{cases} A_s(x, y) = \frac{-20(3x - 2)}{36x^2 - 32x - y}, & B_s(x, y) = \frac{-10(8x + 3y)}{36x^2 - 32x - y}, \\ C_s(x, y) = \frac{3x - 2}{5y(36x^2 - 32x - y)}, & D_s(x, y) = \frac{-198x^2 + 180x + 7y}{5y(36x^2 - 32x - y)}, \\ P_s(x, y) = \frac{-3}{(36x^2 - 32x - y)}, & Q_s(x, y) = \frac{3}{100y(36x^2 - 32x - y)}. \end{cases}$$

5.3 Exact relation between period differential equation unifomizing differential equation

The modular isomorphism (5.1) implies that our period differential equation (3.9) is closely related to the uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$. In this subsection we show that the holomorphic conformal structure of (3.9) is transformed to (5.6) in Theorem 5.1 by an explicit birational transformation. Moreover we determine a normalizing factor which is different from that of Sato's (5.8). The uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ with our normalizing factor corresponds to the period differential equation (3.9).

Proposition 5.4. *The period differential equation (3.9) is represented in the form*

$$\begin{cases} z_{\lambda\lambda} = L_0 z_{\lambda\mu} + A_0 z_\lambda + B_0 z_\mu + P_0 z, \\ z_{\mu\mu} = M_0 z_{\lambda\mu} + C_0 z_\lambda + D_0 z_\mu + Q_0 z \end{cases} \quad (5.9)$$

with

$$\begin{cases} L_0 = \frac{2\mu(-1 + 15\lambda + 100\lambda^2)}{\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu}, & M_0 = \frac{2(\lambda^2 - 8\lambda^3 + 16\lambda^4 + 5\mu - 50\lambda\mu)}{\mu(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu)}, \\ A_0 = \frac{(-1 + 10\lambda)(1 + 20\lambda)}{\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu}, & B_0 = \frac{5\mu(3 + 40\lambda)}{\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu}, \\ C_0 = -\frac{5(-1 + 10\lambda)}{\mu(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu)}, & D_0 = \frac{-\lambda - 20\lambda^2 + 96\lambda^3 - 200\mu}{\mu(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu)}, \\ P_0 = \frac{2(1 + 20\lambda)}{\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu}, & Q_0 = -\frac{10}{\mu(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu)}. \end{cases}$$

Proof. Straightforward calculation. □

Especially the holomorphic conformal structure of the period differential equation (3.9) is

$$\frac{2\mu(-1 + 15\lambda + 100\lambda^2)}{\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu} (d\lambda)^2 + 2(d\lambda)(d\mu) + \frac{2(\lambda^2 - 8\lambda^3 + 16\lambda^4 + 5\mu - 50\lambda\mu)}{\mu(\lambda + 16\lambda^2 - 80\lambda^3 + 125\mu)} (d\mu)^2. \quad (5.10)$$

Theorem 5.2. *Set*

$$f : (\lambda, \mu) \mapsto (x, y) = \left(\frac{25\mu}{2(\lambda - 1/4)^3}, -\frac{3125\mu^2}{(\lambda - 1/4)^5} \right). \quad (5.11)$$

f is a birational transformation from (λ, μ) -space to (x, y) -space. The holomorphic conformal structure (5.10) is transformed to the holomorphic conformal structure (5.6) by *f*.

Proof. f^{-1} is given by

$$\lambda(x, y) = \frac{1}{4} - \frac{y}{20x^2}, \quad \mu(x, y) = -\frac{y^3}{10^5 x^5}. \quad (5.12)$$

We have

$$\begin{cases} L_0(\lambda(x, y), \mu(x, y)) = \frac{-y^2(4x^2 - y)(9x^2 - y)}{250x^3(240x^4 - 88x^2y + 8y^2 - xy^2)}, \\ M_0(\lambda(x, y), \mu(x, y)) = \frac{-4000x^3(100x^4 - 40x^2y + 3x^3y + 4y^2 - xy^2)}{y^2(240x^4 - 88x^2y + 8y^2 - xy^2)}. \end{cases} \quad (5.13)$$

By (5.11), (5.12) we have

$$\begin{cases} x_\lambda = \frac{60x^3}{y}, & y_\lambda = 100x^2, \\ x_\mu = -\frac{10^5 x^6}{y^3}, & y_\mu = -\frac{2 \cdot 10^5 x^5}{y^2}. \end{cases} \quad (5.14)$$

From (5.13), (5.14) and Proposition 5.2, by the birational transformation $f : (\lambda, \mu) \mapsto (x, y)$ the coefficients L_0, M_0 are transformed to

$$\overline{L_0} = \frac{-20(4x^2 + 3xy - 4y)}{36x^2 - 32x - y}, \quad \overline{M_0} = \frac{-2(54x^3 - 50x^2 - 3xy + 2y)}{5y(36x^2 - 32x - y)}.$$

These are equal to the coefficients of the holomorphic conformal structure (5.6). Therefore the holomorphic conformal structure (5.10) is transformed to (5.6). \square

Remark 5.3. *The birational transformation (5.11) is obtained as the composition of certain birational transformations. First blow up at $(\lambda, \mu) = (1/4, 0) \in ((\lambda, \mu)\text{-space})$ 3 times: $(\lambda, \mu) \mapsto (\lambda, u_1) = \left(\lambda, \frac{\mu}{\lambda - 1/4}\right)$, $(\lambda, u_1) \mapsto (\lambda, u_2) = \left(\lambda, \frac{u_1}{\lambda - 1/4}\right)$, $(\lambda, u_2) \mapsto (\lambda, u_3) = \left(\lambda, \frac{u_2}{\lambda - 1/4}\right)$. Then cancel λ by $\lambda = \frac{u_2}{u_3} + \frac{1}{4}$. We have the following birational transformation:*

$$\psi_0 : (\lambda, \mu) \mapsto (u_2, u_3) = \left(\frac{\mu}{(\lambda - 1/4)^2}, \frac{\mu}{(\lambda - 1/4)^3} \right).$$

(Its inverse is

$$\psi_0^{-1} : (u_2, u_3) \mapsto (\lambda, \mu) = \left(\frac{u_2}{u_3} + \frac{1}{4}, \frac{u_2^3}{u_3^3} \right).$$

On the other hand, blow up at $(x, y) = (0, 0) \in ((x, y)\text{-space})$:

$$\psi_1 : (x, y) \mapsto (x, s) = \left(x, \frac{y}{x} \right).$$

(Its inverse is given by

$$\psi_1^{-1} : (x, s) \mapsto (x, y) = (x, xs).$$

Moreover we define the holomorphic map

$$\chi : (u_2, u_3) \mapsto (x, s) = \left(\frac{25}{2}u_3, -250u_2 \right).$$

We see $f = \psi_1^{-1} \circ \chi \circ \psi_0$.

We need the following uniformizing differential equation for our further discussion. This is different from Sato's equation referred in Remark 5.2.

Proposition 5.5. *The uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H}) / \langle PSL(2, \mathcal{O}), \tau \rangle$ with the normalization factor*

$$e^{2\theta} = \frac{x^4(-36x^2 + 32x + y)}{y^{5/2}(1728x^5 - 720x^3y + 80xy^2 - 64(5x^2 - y)^2 - y^3)^{3/2}} \quad (5.15)$$

is

$$\begin{cases} z_{xx} = L_1 z_{xy} + A_1 z_x + B_1 z_y + P_1 z, \\ z_{yy} = M_1 z_{xy} + C_1 z_x + D_1 z_y + Q_1 z \end{cases} \quad (5.16)$$

with

$$\begin{cases} L_1 = \frac{-20(4x^2 + 3xy - 4y)}{36x^2 - 32x - y}, & M_1 = \frac{-2(54x^3 - 50x^2 - 3xy + 2y)}{5y(36x^2 - 32x - y)}, \\ A_1 = \frac{-2(20x^3 - 8xy + 9x^2y + y^2)}{xy(36x^2 - 32x - y)}, & B_1 = \frac{10y(-8 + 3x)}{x(36x^2 - 32x - y)}, \\ C_1 = \frac{-2(-25x^2 + 27x^3 + 2y - 3xy)}{5y^2(36x^2 - 32x - y)}, & D_1 = \frac{-2(-120x^2 + 135x^3 - 2y - 3xy)}{5xy(36x^2 - 32x - y)}, \\ P_1 = \frac{-2(8x - y)}{25x(36x^2 - 32x - y)}, & Q_1 = \frac{-2(-10 + 9x)}{25xy(36x^2 - 32x - y)}. \end{cases}$$

Proof. L_1 and M_1 is given in Theorem 5.1. According to Proposition 5.1 the other coefficients are determined by L_1, M_1 and θ in (5.15). \square

Theorem 5.3. *By the birational transformation f in (5.11), the equation (5.9) is transformed to the equation (5.16).*

Proof. We have

$$\left\{ \begin{array}{l} A_0(\lambda(x, y), \mu(x, y)) = \frac{400x^2(3x^2 - y)(6x^2 - y)}{y(240x^4 - 88x^2y + 8y^2 - xy^2)}, \\ B_0(\lambda(x, y), \mu(x, y)) = \frac{-y^2(13x^2 - 2y)}{25x(240x^4 - 88x^2y + 8y^2 - xy^2)}, \\ C_0(\lambda(x, y), \mu(x, y)) = \frac{2 \cdot 10^8 x^9(3x^2 - y)}{y^4(240x^4 - 88x^2y + 8y^2 - xy^2)}, \\ D_0(\lambda(x, y), \mu(x, y)) = \frac{160000x^5(175x^4 - 65x^2y + 6y^2 - xy^2)}{y^3(240x^4 - 88x^2y + 8y^2 - xy^2)}, \\ P_0(\lambda(x, y), \mu(x, y)) = \frac{1600x^4(6x^2 - y)}{y(240x^4 - 88x^2y + 8y^2 - xy^2)}, \\ Q_0(\lambda(x, y), \mu(x, y)) = \frac{8 \cdot 10^8 x^{11}}{y^4(240x^4 - 88x^2y + 8y^2 - xy^2)}. \end{array} \right. \quad (5.17)$$

By (5.11), (5.12) we have

$$\left\{ \begin{array}{l} x_{\lambda\lambda} = \frac{4800x^5}{y^2}, \quad y_{\lambda\lambda} = \frac{12000x^4}{y}, \\ x_{\mu\mu} = 0, \quad y_{\mu\mu} = \frac{2 \cdot 10^{10}x^{10}}{y^5}, \\ x_{\lambda\mu} = \frac{-6 \cdot 10^6x^8}{y^4}, \quad y_{\lambda\mu} = \frac{-2 \cdot 10^7x^7}{y^3}. \end{array} \right. \quad (5.18)$$

From (5.13), (5.14), (5.17), (5.18) and Proposition 5.2, by the birational transformation $f : (\lambda, \mu) \mapsto (x, y)$ the coefficients $A_0, B_0, C_0, D_0, P_0, Q_0$ is transformed to

$$\left\{ \begin{array}{l} \overline{A_0} = \frac{-2(20x^3 - 8xy + 9x^2y + y^2)}{xy(36x^2 - 32x - y)}, \quad \overline{B_0} = \frac{10y(-8 + 3x)}{x(36x^2 - 32x - y)}, \\ \overline{C_0} = \frac{-2(-25x^2 + 27x^3 + 2y - 3xy)}{5y^2(36x^2 - 32x - y)}, \quad \overline{D_0} = \frac{-2(-120x^2 + 135x^3 - 2y - 3xy)}{5xy(36x^2 - 32x - y)}, \\ \overline{P_0} = \frac{-2(8x - y)}{25x(36x^2 - 32x - y)}, \quad \overline{Q_0} = \frac{-2(-10 + 9x)}{25xy(36x^2 - 32x - y)}. \end{array} \right.$$

These are equal to the coefficients of (5.16). \square

Therefore the uniformizing differential equation of the orbifold $(\mathbb{H} \times \mathbb{H})/\langle PSL(2, \mathcal{O}), \tau \rangle$ with the normalization factor (5.15) is equal to the period differential equation of the family \mathcal{F} of $K3$ surfaces $S(\lambda, \mu)$.

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References

- [Ba] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geometry **3**, 1994, 493-535.
- [Be] S. M. Belcastro, *Picard lattices of families of K3 surfaces*, Comm. Algebra **30** (1), 2002, 61-82.
- [H] F. Hirzebruch, *The ring of Hilbert modular forms for real quadratic fields of small discriminant*, Lecture Notes in Math. **627**, Springer-Verlag, 1977, 287-323.
- [I1] T. Ishige, *An isomorphic correspondence between the Hilbert modular group and the unimodular group of a domain of Type IV*, Technical Report, Chiba Univ., 2008.
- [I2] T. Ishige, *A Family of K3 Surfaces connected with the Hilbert Modular Group for $\sqrt{2}$ and the GKZ Hypergeometric Differential Equation*, preprint, 2010.
- [KKN] R. Kobayashi, K. Kushibiki and I. Naruki, *Polygons and Hilbert modular groups*, Tohoku Math. **41**, 1989, 633-646.
- [Kod] K. Kodaira, *On compact analytic surfaces. II*, Ann. Math., **77**, 1963, 563-626.
- [Koi] K. Koike, *K3 surfaces induced from polytopes*, Master Thesis, Chiba Univ., 1998.
- [KSTT] K. Koike, H. Shiga, N. Takayama and T. Tsutsui, *Study on the family of K3 surfaces induced from the lattice $(D_4)^3 \oplus \langle -2 \rangle \oplus \langle 2 \rangle$* , Internat. J. Math., **12**, No. 9, 2001, 1049-1085.
- [KS] M. Kreuzer and H. Sharke, *Classification of Reflexive Polyhedra in Three Dimensions*, Adv. Theor. Math. Phys., **2**, 847-864.
- [Ma] K. Matsumoto, *A system of generators of $O_H(\mathbb{Z})$* , preprint, 2009.
- [MSY] K. Matsumoto, T. Sasaki and M. Yoshida, *The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type $(3,6)$* , Internat. J. Math. **3**, 1992, 1-164.
- [Mo] D. R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75**, 1984, 105-121.
- [N] A. Nagano, *Period differential equations for families of K3 surfaces derived from reflexive polytopes*, preprint, 2010.
- [NS] N. Narumiya and H. Shiga, *The Mirror Map for a Family of K3 Surfaces induced from the Simplest 3-Dimensional Reflexive Polytope*, CNR Proceedings and Lecture Notes Vol. 30, 2001.
- [N] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv., **14**, 1980, No.1, 103-167.
- [O] M. Otsuka, *3-dimensional reflexive polytopes*, Master Thesis, Chiba Univ., 1998.
- [PS] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR Izv. **5**(3), 1971, 547-588.
- [SY] T. Sasaki and M. Yoshida, *Linear differential equations in two variables of rank four I-II*, Math. Ann. **282**, 1988, 69-111.
- [Sa] T. Sato, *Uniformizing differential equations of several Hilbert modular orbifolds*, Math. Ann. **291**, 1991, 179-189.

- [Sh1] H. Shiga, *One Attempt to the K3 modular function I*, Ann. Scuola Norm. Pisa, Serie IV - Vol. VI, 1979, 609-635.
- [Sh2] H. Shiga, *One Attempt to the K3 modular function II*, Ann. Scuola Norm. Pisa, Serie IV - Vol. VIII, 1981, 157-182.
- [Sh3] H. Shiga, *One Attempt to the K3 modular function III*, Technical Report, Chiba Univ., 2010.
- [St] J. Stienstra, *A^1 -hypergeometric functions, A -discriminants and Fiber polytopes*, seminar on hypergeometric functions Utrecht, February 23 and March 2, 1993.
- [TN] N. Takayama and M. Nakayama, *Calculations of hypergeometric differential equations connected with Fano polytopes*, preprint, 2009.

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